Übungsblatt 08

Aufgabe 1

(a) Normierungsbedingung:

$$1 = \int_0^{2\pi} \Psi^*(\varphi)\Psi(\varphi)\,d\varphi$$

$$1 = N^2 \int_0^{2\pi} e^{-im_e\varphi} e^{im_e\varphi}\,d\varphi = N^2 \int_0^{2\pi} d\varphi = N^2 2\pi$$

$$N = \frac{1}{\sqrt{2\pi}}$$

(b) Orthogonalitätsbedingung für die beiden Funktionen $\Psi_1(\varphi)$ und $\Psi_2(\varphi)$ mit den Quantenzahlen $m_{e,1}$ und $m_{e,2}$:

$$0 = \int_0^{2\pi} \Psi_1^*(\varphi)\Psi_2(\varphi)\,d\varphi$$

$$N^2 \int_0^{2\pi} e^{-im_{e,1}\varphi} e^{im_{e,2}\varphi}\,d\varphi = N^2 \int_0^{2\pi} e^{(m_{e,2}-m_{e,1})i\varphi}\,d\varphi =$$

$$= /\ k = m_{e,2} - m_{e,1};\ k \in N\ / =$$

$$= N^2 \int_0^{2\pi} e^{ik\varphi}\,d\varphi = N^2 \left(-\frac{i}{k} e^{ik\varphi}\right)\Big|_0^{2\pi} = N^2\left[-\frac{i}{k} e^{ik2\pi} + \frac{i}{k} e^{ik0}\right] =$$

$$= N^2\left[-\frac{i}{k}\left(\cos(k2\pi) + i\sin(k2\pi)\right) + \frac{i}{k}\right] =$$

$$= N^2\left[-\frac{i}{k} + \frac{i}{k}\right] = 0$$

Aufgabe 2

$$L^2 \Psi = \hbar^2 \lambda \Psi$$

$$L_z \Psi = m\hbar \Psi$$

$$L_z^2 \Psi = L_z(L_z \Psi) = L_z(m\hbar \Psi) = m\hbar L_z \Psi = m^2\hbar^2 \Psi$$

$$L_x^2 + L_y^2 = L^2 - L_z^2$$

$$(L_x^2 + L_y^2)\Psi = (L^2 - L_z^2 \Psi) = L^2\Psi - L_z^2 \Psi =$$

$$= \hbar^2 \lambda \Psi - m^2\hbar^2 \Psi = (\hbar^2\lambda - \hbar^2 m^2)\Psi = \hbar^2(\lambda - m^2)\Psi$$

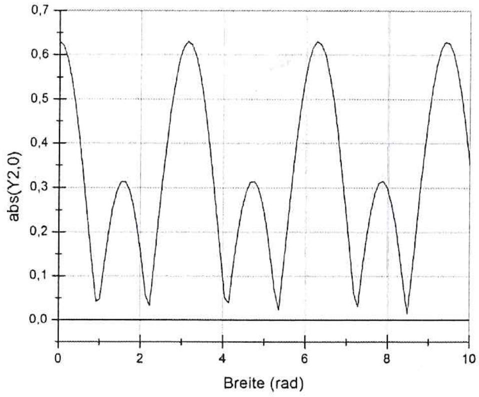
Ja, $Y_{l,m}$ sind auch Eigenfunktionen von $L_x^2 + L_y^2$

Eigenwerte sind $\hbar^2(\lambda - m^2)$

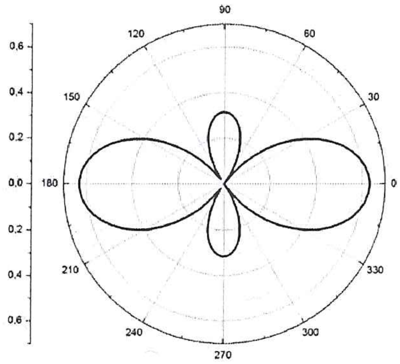
Aufgabe 3

$$Y_{2,0}(\theta) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

In kartesischen Koordinaten:



In Polarkoordinaten:



Aufgabe 4

(a) Normierungsbedingung:

$$1 = \int_0^{2\pi}\int_0^{\pi} |Y_{2,0}|^2 \sin\vartheta \, d\vartheta d\varphi =$$

$$= 2\pi \int_0^{\pi} |Y_{2,0}|^2 \sin\vartheta d\vartheta =$$

$$= 2\pi C^2 \int_0^{\pi} (3\cos^2\vartheta - 1)^2 \sin\vartheta d\vartheta = / \sin\vartheta \, d\vartheta = - d\cos\vartheta / =$$

$$= 2\pi C^2 \int_{-1}^{1} (3\cos^2\vartheta - 1)^2 d\cos\vartheta = / x = \cos\vartheta / =$$

$$= 2\pi C^2 \int_{-1}^{1} (3x^2 - 1)^2 dx =$$

$$= 2\pi C^2 \int_{-1}^{1} (9x^4 - 6x^2 + 1) dx = 2\pi C^2 \left(\frac{9}{5}x^5 - \frac{6}{3}x^3 + x \right)\Big|_{-1}^{1} =$$

$$= 2\pi C^2 \left(\frac{9}{5}2 - \frac{6}{3}2 + 2 \right) = 2\pi C^2 \left(\frac{18}{5} - 2 \right) = \frac{16}{5}\pi C^2 = 1$$

$$C = \sqrt{\frac{5}{16\pi}}$$

(B) $Y_{2,0} \sim (3\cos^2\theta - 1)$

$$Y_{2,+1} = L_+ Y_{2,0} = C(L_x + iL_y)(3\cos^2\theta - 1)\,.$$

$$L_x = -\frac{\hbar}{i}\left\{\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$L_y = \frac{\hbar}{i}\left\{\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$\frac{\partial}{\partial\varphi}(3\cos^2\theta - 1) = 0$$

$$Y_{2,+1} = C\left[-\frac{\hbar}{i}\sin\varphi\frac{\partial}{\partial\theta}(3\cos^2\theta - 1) + i\frac{\hbar}{i}\cos\varphi\frac{\partial}{\partial\theta}(3\cos^2\theta - 1)\right] =$$

$$= C\left[-\frac{\hbar}{i}\sin\varphi\frac{\partial}{\partial\theta}3\cos^2\theta + \hbar\cos\varphi\frac{\partial}{\partial\theta}3\cos^2\theta\right] =$$

$$= C[-i\hbar\sin\varphi\, 6\cos\theta\sin\theta - \hbar\cos\varphi\, 6\cos\theta\sin\theta] =$$

$$= -6C\hbar\cos\theta\sin\theta[\cos\varphi + i\sin\varphi] = -6C\hbar\cos\theta\sin\theta e^{i\varphi} =$$

$$= C_1\cos\theta\sin\theta e^{i\varphi}$$

$$L_+ = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]$$

$$Y_{2,+2} = L_+ Y_{2,+1} = C_1\hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]\frac{1}{2}\sin 2\theta e^{i\varphi} =$$

$$= \frac{C_1}{2}\hbar e^{i\varphi}\left[2\cos 2\theta e^{i\varphi} + i\cot\theta\sin 2\theta\frac{\partial}{\partial\varphi}e^{i\varphi}\right] =$$

$$= \frac{C_1}{2}\hbar e^{i\varphi}\left[2\cos 2\theta e^{i\varphi} + i\,2\cos^2\theta \cdot i e^{i\varphi}\right] =$$

$$= 2\frac{C_1}{2}\hbar e^{2i\varphi}\left[\cos^2\theta - \sin^2\theta - \cos^2\theta\right] = -C_1\hbar e^{2i\varphi}\sin^2\theta =$$

$$= -C_1\hbar\sin^2\theta e^{2i\varphi}$$